Falaco Solitons - Cosmic strings in a swimming pool.

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Abstract: A dynamical bifurcation mechanism related to changes of metric signature is used to explain the formation of topological defects experimentally observed and defined in 1986 as Falaco Solitons. The Falaco Solitons are topologically coherent structures created experimentally by a macroscopic rotational dynamics in a continuous media with a discontinuity surface, such as that found in a swimming pool. The topological structure of Falaco Solitons replicates certain features found at all physical scales, from spiral arm galaxies and cosmic strings to submicroscopic hadrons. The easy to replicate experiment indicates that the creation of "stationary" thermodynamic states (or solitons) far from equilibrium can be globally stabilized. The Falaco solitons represent a paradigm for explaining a spin pairing mechanism in the microscopic Fermi surface, the development of dimpled vortex structures in rotating Bose-Einstein Condensates, the confinement problem of sub-microscopic quarks on the end of a string connecting branes, and the quantized needle radiation of as photon in terms of a string connecting two Falaco Soliton dimples located on concentric light cone shells.

Keywords: Falaco Solitons, Hopf Breathers, Cosmic Strings between Galaxies, Global Stability, Rotating Bose-Einstein condensates, Bifurcation between Minkowski and Euclidean domains.

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1. Falaco Solitons - What are they?

1.1. A Topologically Coherent Fluid Defect.

During March of 1986, while visiting an old MIT friend in Rio de Janeiro, Brazil, the present author became aware of a significant topological event involving visual solitons that can be replicated experimentally by almost everyone with access to a swimming pool. Study the photo which was taken by David Radabaugh, in the late afternoon, Houston, TX 1986.

Figure 1. Falaco Solitons in a Swimming Pool

The extraordinary photo is an image of 3 pairs of what are now called Falaco Solitons, a few minutes after their creation. Each Falaco Soliton consists of a pair of globally stabilized rotational indentations in the water-air discontinuity surface of the swimming pool. The dimple shape is as if a conical pencil point was pushed into a rubber sheet causing a deformation, but the indentation is dominated by dynamic rotation, not translation. Unseen in the photograph, each pair of contra-rotating dimples are connected by a singular thread in the form of a circular arc extending from the vertex of one dimple to the vertex of the other dimple of the pair. The "thread" can be made visible by injecting drops of dye into the fluid near the rotation axis of one of the dimples. These Solitons are apparently long-lived states of matter far from thermodynamic equilibrium. They will persist for many minutes in a still pool of water, maintaining their topological coherence so as to permit their inclusion into the class of objects called Solitons. The Falaco Solitons are extraordinary, not only due to the fact that they are so easily created in a macroscopic dynamical systems environment, but also because they offer real life, easily observed, evidence for the continuous evolution and creation of topological defects.

The long lifetime, and the topological stability, of the Falaco Solitons in a dissipative fluid media is not only remarkable but also is a matter of applied theoretical interest. The equilibrium discontinuity surface of the fluid in the "uniform" g field is flat, and has both zero mean curvature and zero Gauss curvature. The shape of the observed discontinuity surface defect of a Falaco Soliton dimple indicates that the surface mean curvature is zero, but the Gauss curvature is not zero. In Euclidean spaces, such real surfaces are *minimal* surfaces of negative Gauss curvature. Such surfaces are locally unstable, so it has been presumed that the pair of defect structures that make up the Falaco Soliton must be globally stabilized. It has been conjectured that the connecting string is under tension in order to maintain the shape of the pair of dimpled indentations. This conjecture is justified by the observation that if the singular thread is abruptly "severed" (by experimental chopping motions under the surface of the fluid), the dimpled endcaps disappear in a rapid, non-diffusive, manner.

The dimpled surface pairs of the Falaco Soliton are most easily observed in terms of the dramatic black discs that they create by projection of the solar rays to the bottom of the pool. The optics of this effect will be described below. Careful examination of the photo of Figure 1 will indicate, by accidents of noticeable contrast and reflection, the region of the dimpled surface of circular rotation. The dimples appear as (deformed) artifacts to the left of each black spot, and elevated above the horizontal plane by about 25 degrees (as the photo was taken in late afternoon). Also, notice that the vestiges of caustic spiral arms in the surface structures around each pair of rotation axes can be seen. These surface spiral arms can be visually enhanced by spreading chalk dust on the free surface of the pool. The bulk fluid motion is a local (non-rigid body) rotational motion about the interconnecting circular thread. In the photos of Figure 1 and Figure 2, the depth of each of the actual indentations of the free surface is, at most, of a few millimeters in extent.

A better photo, also taken by D. Radabaugh, but in the year 2004 in a swimming pool in Mazan, France, demonstrates more clearly the dimpled surface defects, and the Snell refraction. The sun is to the left and at an elevation of about 30 degrees.

Figure 2. Surface Indentations of a Falaco Soliton

The photo is in effect a single frame of a digital movie that demonstrates the creation and evolutionary motions of the Falaco Solitons. The experimental details of creating the Falaco Solitons are described below, but the movie explains their creation and dynamics far better than words. The digital movie may be downloaded from [23].

Remark 1. *The bottom line is that it is possible to produce, hydrodynamically, in a viscous fluid with a surface of discontinuity, a long lived topologically coherent structure that consists of a set of macroscopic topological defects. The Falaco Solitons are representative of non-equilibrium long lived structures, or "stationary states", far from equilibrium.*

These observations were first reported at the 1987 Dynamics Days conference in Austin, Texas [14] and subsequently in many other places, mostly in the hydrodynamic literature [15], [16], [20], [22], as well as several APS meetings. More detail is presented in [29].

1.2. Falaco Surface dimples are of zero mean curvature

From a mathematical point of view, the Falaco Soliton is interpreted as a connected pair of two dimensional topological defects connected by a one dimensional topological defect or thread. The surface defects of the Falaco Soliton are observed dramatically due the formation of circular black discs on the bottom of the swimming pool. The very dark black discs are emphasized in contrast by a bright ring or halo of focused light surrounding the black disc. All of these visual effects can be explained by means of the unique optics of Snell refraction from a surface of zero mean curvature.

Remark 2. *This explanation of the optics was reached about 30 minutes after I first became aware of the Soliton structures, while standing in the pristine white marble swimming pool of an old MIT roommate, Jose Haraldo Falçao, under the brilliant Brazilian sunshine in Rio de Janeiro. At MIT, Haraldo was always called Falaco, after he scored 2 goals in a MIT soccer match, and the local newspapers misprinted his name. Hence I dubbed the topological defect structures, Falaco Solitons. Haraldo will get his place in history. I knew that finally I had found a visual, easily reproduced, experiment that could be used to show people the importance and utility of Topological Defects in the physical sciences, and could be used to promote my ideas of Continuous Topological Evolution.*

The observations were highly motivating. The experimental observation of the Falaco Solitons greatly stimulated me to continue research in applied topology, involving topological defects, and the topological evolution of such defects which can be associated with phase changes and thermodynamically irreversible and turbulent phenomena. When colleagues in the physical and engineering sciences would ask "What is a topological defect?" it was possible for me to point to something that they could replicate and understand visually at a macroscopic level.

During the initial few seconds of decay to the metastable soliton state, each large black disk is decorated with spiral arm caustics, remindful of spiral arm galaxies. The spiral arm caustics contract around the large black disk during the stabilization process, and ultimately disappear when the "topological steady" soliton state is achieved. The spiral caustics appear to be swallowed up by the black "hole". It should be noted that if chalk dust is sprinkled on the surface of the pool during the formative stages of the Falaco Soliton, then the topological signature of the familiar Mushroom Spiral pattern is exposed.

Notice that the black spots on the bottom of the pool in the photo are circular and not distorted ellipses, even though the solar elevation is less than 30 degrees. The important experimental fact deduced from the optics of Snell refraction is that each dimpled surface appears to be a surface of zero mean curvature. This conclusion is justified by the fact that the Snell projection to the floor of the pool is almost conformal, preserving the circular appearance of the black disc, independent from the angle of solar incidence. This conformal projection property of preserving the circular shape is a property of normal projection from minimal surfaces of zero mean curvature [24].

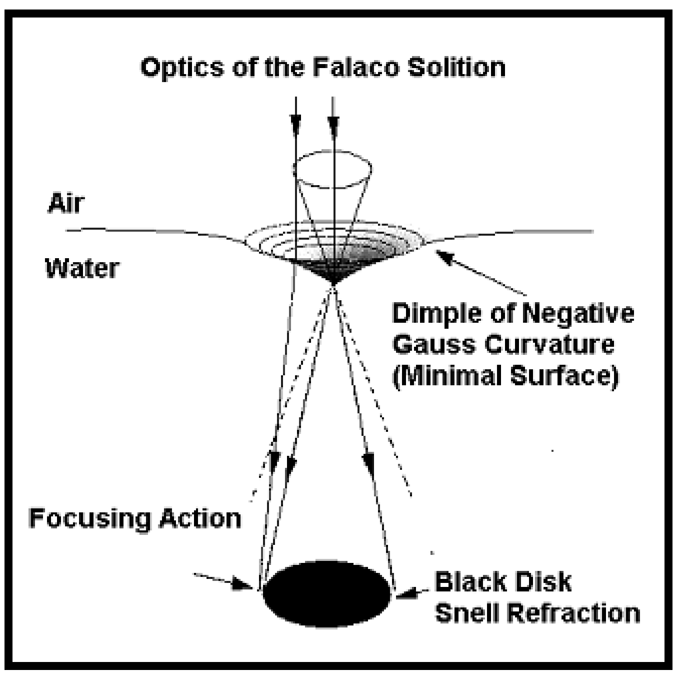


Figure 3. Snell Refraction of a Falaco Soliton surface defect.

As mentioned above, a feature of the Falaco Soliton [14] that is not immediately obvious is that it consists of a pair of two dimensional topological defects, in a surface of fluid discontinuity, which are *connected* by means of a topological singular thread. Dye injection near an axis of rotation during the formative stages of the Falaco Soliton indicates that there is a unseen thread, or 1-dimensional string singularity, in the form of a circular arc that connects the two 2-dimensional surface singularities or dimples. Transverse Torsional waves made visible by dye streaks (caused by dye drops injected near one of the surface rotation axes) can be observed to propagate, back and forth, from one dimple vertex to the other dimple vertex, guided by the "string" singularity. The effect is remindful of the whistler propagation of electrons along the guiding center of the earth's pole to pole magnetic field lines.

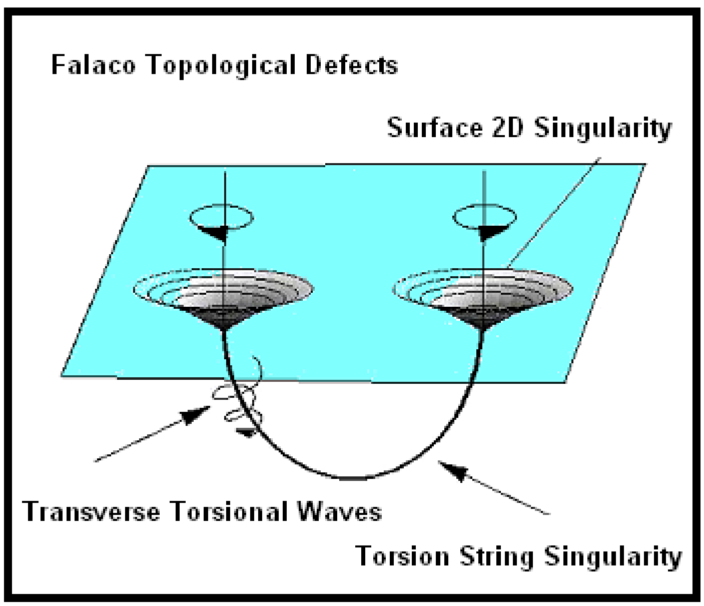


Figure 4. Falaco Topological Defects with connecting thread.

However, as a soliton, the topological system retains its coherence for remarkably long time - more than 15 minutes in a still pool. The long lifetime of the Falaco Soliton is conjectured to be due to this *global stabilization* of the connecting string singularity, even though a real surface of zero mean curvature is locally unstable. The Falaco Soliton starts out from a non-equilibrium thermodynamic state of Pfaff topological dimension 4, which quickly and irreversibly decays to a "topologically stationary" state, still far from equilibrium, but with a long dynamic lifetime [28] [29].

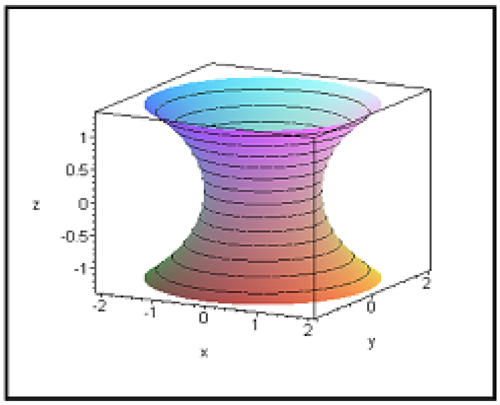
1.3. Falaco Surfaces are related to Harmonic vector fields.

The long life of the soliton state in the presence of a viscous media indicates that the flow vector field describing the dynamics is probably harmonic. This result is in agreement with the assumption that the fluid can be represented by a Navier-Stokes equation where the viscous dissipation is dominated by affine shear viscosity times the vector Laplacian of the velocity field. If the velocity field is harmonic, the vector Laplacian vanishes, and the shear dissipation term goes to zero - no matter what is the magnitude of the shear viscosity term. Hence a palatable argument is offered in terms of harmonic velocity fields for the existence of the long lifetime of the Falaco Solitons (as well as the production of wakes in fluid dynamics [30]). More over it is known in the theory of minimal surfaces [12] that surfaces of zero mean curvature are generated by harmonic vector fields.

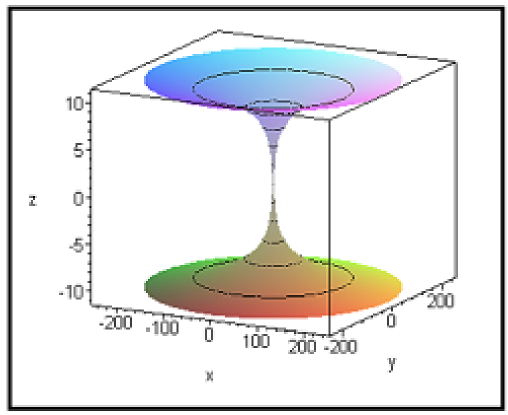
Remark 3. *The bottom line is that the idea of a long lifetime in a dissipative media is to be associated with Harmonic vector fields and surfaces of zero mean curvature.*

Initially it was thought by the present author that the surface configuration, immediately after creation, was in the form of a Rankine vortex (of positive mean curvature, and positive Gauss curvature in a 3D euclidean space), which then decayed into a classic *minimal* surface of zero mean curvature, but negative Gauss curvature. Such an evolutionary process can be found in Langford bifurcations [29] which can be shown to be solutions to the Navier-Stokes equations in a rotating frame of reference. However, such a dynamics seems to require that the connection (the string) between the Falaco pairs has an open throat (like a Wheeler worm hole). Note that for a "stationary" euclidean soap film between two boundary rings (Fig. 5a), the system is stable only if the separation of the boundary rings is less than (approximately) 2.65 times the minimal throat diameter. Experimentally the *stationary non-rotating* soap film between two boundary rings will break apart if the soap film is stretched too far. The single component catenoid (with zero mean curvature and negative Gauss curvature, and with real equal and opposite principle curvatures) will bifurcate into two flat components, one on each ring, and each of zero Gauss curvature as well as zero mean curvature. The process has been demonstrated in fluid flow in a rotating frame, where the zero helicity function of the fluid flow has the appearance of a minimal surface. As the bulk flow increases, the helicity function changes sign, and therefor represents a change in topology from a connected set to a disconnected set. With the change in sign, a torsion bubble (or a torsion burst) appears in the flow pattern [16] [30].

Admittedly, the extended catenoid of Figure 5b (a deformed Wheeler Wormhole with an open throat ?) has the some of the features and appearance of the Falaco Solitons, but the extended singular thread (without an open throat) between vertex singularities does not appear to be replicated.



Soap film between rings
Figure 5a.



Deformed Wheeler Wormhole
Figure 5b.

It remains difficult to utilize the minimal surface soap film conjecture of a decaying Rankine vortex to support the idea of topological evolution to a structure of two dimpled surfaces of zero mean curvature connected by a 1 dimensional thread. The question arises as how to explain the creation and existence of the Falaco Solitons. The idea that the Falaco Solitons are related to strings connecting branes led to the thought that perhaps the modern advances in topology and string theory could yield a theoretical explanation. According, challenges and requests for help were sent out to many of the string theorists, asking for theoretical help to describe this "real life string connecting branes"; the lack of response indicates that none of the string gurus seemed to think the effort was worthwhile. However, the theoretical work of Dzhunushaliev [4] seems to have many correspondences with the experimental facts of the Falaco Solitons.

In Euclidean space, the real minimal surface defects of zero mean curvature are of negative (or zero) Gauss curvature, and are, therefor, locally unstable. However, stationary non rotating soap films can be stabilized by certain boundary conditions. As mentioned above the experimental equilibrium state of the fluid discontinuity surface is a surface of both zero Gauss curvature and zero Mean curvature (both principle surface curvatures are zero). From the optics of Snell refraction, a Falaco endcap is obviously a surface of zero mean curvature, and if equivalent to a stationary soap film, it should be locally unstable. However, it was conjectured that the local instability could be overcome globally by a string whose tension globally stabilizes the locally unstable endcaps. Could the tension be related to a rotationally induced positive contribution to the otherwise negative Gauss curvature? These conjectures originally were explained (partially) in terms of a bifurcation process and solutions to the Navier-Stokes equations in a rotating frame of reference [29]. A summary of such analysis is presented below.

More recently, it was determined that an alternative, and perhaps better, description can be given in terms of a fluid with a surface discontinuity that has zero mean curvature relative to a Minkowski metric; the Minkowski surface has a Gauss curvature which is positive. A topological birfurcation process from a Rankine vortex to Falaco Solitons would then be such as to change the 3D Euclidean signature into a 3D Minkowski signature. Such surfaces of zero mean curvature embedded in Minkowski space have been called *maximal* surfaces by the differential geometers, and have conical singular points [5] . It is now believed that the Falaco thread is attached to the conical singular points of a pair of such maximal surfaces.

Alternatively, the Euclidean metric can be maintained, and a result similar to the immersion of the 2D surface into Minkowski space can be attributed to the fact that infinitesimal rotations admit Spinor complex isotropic eigen direction fields with non-zero, pure imaginary eigen values. The Gauss curvature of such systems is positive, even though the eigen direction fields are complex Spinors, not vectors in the diffeomorphic sense. A discussion of the Hopf map as applied to this idea will be found below.

1.4. Spinors and zero mean curvature surfaces.

The theory of minimal surfaces (of zero mean curvature) are intertwined with the concept of complex isotropic direction fields, defined as pure Spinors by E.Cartan. The Weirstrass formulas of minimal surface theory [12] consider a holomorphic complex velocity field in 3D, which upon integration leads to conjugate pairs of minimal surfaces defined by the real and imaginary components of the position vector formed by complex integration. The key feature of this holomorphic "velocity" field, so useful to minimal surface theory, is that it is a complex isotropic collection of components, whose Euclidean sums of squares is zero. Such isotropic complex direction fields of zero quadratic form (length) were defined as Spinors by E. Cartan [2].

In addition, E. Cartan demonstrated that infinitesimal rotations are generated by antisymmetric matrices. It is rather remarkable (and was only fully appreciated by the author in February, 2005) that there is a large class of direction fields (still given the symbol $\rho\mathbf{V}_4$) that do not behave as diffeomorphic vectors. Such direction fields are Spinors and satisfy the equation,

$$\textbf{The Spinor Class:}\quad i(\rho\mathbf{V}_4)dA \neq 0. \tag{1.1}$$

Spinors are eigen direction fields representing processes, for which the eigenvalues are not zero, but for which the quadratic sums of components squared is zero.

To understand these claims, realize that the Work 1-form is a generalization [28] of the Newtonian concept of Force times Distance:

$$W = i(\rho\mathbf{V}_4)dA = f_m dx^m + Pdt \tag{1.2}$$

The 2-form dA can be realized as an anti-symmetric matrix of functions. The concept of Work as the 1-form $W = i(\rho\mathbf{V}_4)dA$ focuses attention on the importance of the 2-form, $F = dA$, and its antisymmetric matrix representation, $F \simeq [\mathbb{F}] = -[\mathbb{F}]^{transpose}$. The concept of Work is (in effect) related to the matrix product of $[\mathbb{F}]$ and some vector direction field:

$$W = i(\rho\mathbf{V}_4)dA \simeq [\mathbb{F}] \circ |\rho\mathbf{V}_4\rangle\,. \tag{1.3}$$

Suppose $\mathbf{e} = \rho\mathbf{V}_4$ is an eigendirection field with eigenvalue γ, such that

$$i(\rho\mathbf{V}_4)dA \simeq [\mathbb{F}] \circ |\mathbf{e}\rangle = \gamma\,|\mathbf{e}\rangle\,. \tag{1.4}$$

Then,

$$i(\rho\mathbf{V})i(\rho\mathbf{V})dA \simeq \langle\mathbf{e}| \circ [\mathbb{F}] \circ |\mathbf{e}\rangle = \gamma\,\langle\mathbf{e}| \circ |\mathbf{e}\rangle\,. \tag{1.5}$$

Due to antisymmetry, it follows that

$$i(\rho\mathbf{V})i(\rho\mathbf{V})dA \simeq \langle\mathbf{e}| \circ [\mathbb{F}] \circ |\mathbf{e}\rangle = 0. \tag{1.6}$$

Hence, for the antisymmetric matrix, $[\mathbb{F}]$, it must be true that

$$\gamma\,\langle\mathbf{e}| \circ |\mathbf{e}\rangle = 0. \tag{1.7}$$

For division algebras there are two choices: either $\gamma = 0$, or $\langle\mathbf{e}| \circ |\mathbf{e}\rangle = 0$. The implication is that for non zero eigenvalues γ, the quadratic form must vanish:

$$\langle \mathbf{e}| \circ |\mathbf{e}\rangle = (e^1)^2 + (e^2)^2 + ... + (e^n)^2 = 0. \tag{1.8}$$

Over the real domain, there are no "real vectors" that satisfy this quadratic form, but there are many complex vectors that satisfy the "isotropic" formula. In Euclidean 3 space, the complex integrals of the complex isotropic vectors, when separated into real and imaginary parts, lead to two conjugate 3D "position vectors" that describe immersions of minimal (zero mean curvature) surfaces in 3D.

Remark 4. *The bottom line is that Falaco Solitons can represent non tensorial properties of Spinor analysis, and, as will be developed below, lead to the possibility of surfaces of zero mean curvature, but with positive, not negative, Gauss curvature.*

1.5. Topological Universality independent from scales.

The reader must remember that the Falaco Soliton is a topological object that can and will appear at all scales, from the microscopic, to the macroscopic, from the sub-submicroscopic world of strings connection branes, to the cosmological level of spiral arm galaxies connected by threads. At the microscopic level, the method offers a view of forming spin pairs that is different from Cooper pairs and could offer insight into Hi-TC Superconductivity. At the level of Cosmology, the concept of Falaco Solitons could lead to explanations of the formation of flat spiral arm galaxies. At the submicroscopic level, the Falaco Solitons mimic quark pairs confined by a string. At the microscopic level, the Falaco Solitons appear as the dimpled vortex structures in rotating Bose-Einstein condensates. They also model the concepts of a Photon as being the singular thread attached to dimples on two expanding light cone shells. At the macroscopic level, similar topological features of the Falaco Solitons can be found in solutions to the Navier-Stokes equations in a rotating frame of reference. Under deformation of the discontinuity surface to a flattened ball, the visual correspondence to hurricane structures between the earth surface and the tropopause is remarkable. In short, as a topological defect, the concept of Falaco Solitons is a universal phenomenon valid at all scales.

1.6. The Experiment

The Falaco Soliton phenomenon is easily reproduced by placing a large circular disc, such as dinner plate, vertically into the swimming pool until the plate is half submerged and it oblate axis resides in the water-air free surface. Then move the plate slowly in the direction of its oblate axis. At the end of the stroke, smoothly extract the plate (with reasonable speed) from the water, imparting kinetic energy and distributed angular momentum to the fluid. The dynamical system undergoes a short period (a few seconds) of stabilization, followed by a longer period (many minutes) of a "topologically stationary" state. It is this topologically stationary state that is defined as the Falaco Soliton. Thermodynamically, the system starts in an initial state of Pfaff topological dimension 4 and decays by continuous topological evolution to a "stationary" state of Pfaff topological dimension 3. According to the theory of non equilibrium thermodynamics [28], the processes during the initial stabilization period are thermodynamically irreversible, but once the Pfaff dimension 3 configuration is reached, the evolutionary processes preserving topological features can be described in a Hamiltonian manner. Both the initial and the "stationary" soliton states are thermodynamic states far from equilibrium.

At first it was thought that the initial deformed surface state could be related to a Rankine vortex structure (which has regions of both positive and negative Gauss curvature). Recall that a Rankine vortex has a core that is equivalent to rigid body rotation. This description of the formative state of stabilization is too naive, for observations indicate that the sharp edge of the plate described above generates instability patterns [30] as it is stroked through the fluid. After the initial injection of energy and angular momentum, the fluid spends a few seconds during a process of stabilization, during which a surface of zero mean curvature is formed transiently, producing the easily visible large black spots formed by Snell refraction. Associated with the evolution to a "stationary" Soliton state, is a visible set of spiral arm caustics on the pool surface around each dimples rotation axis. As the stabilization proceeds, the spiral caustics appear to grow tighter around the black spot, and are almost gone when the Soliton becomes stable.

In a few tries you will become an expert experimentalist at stroking the plate and creating Falaco Solitons. The drifting black spots are easily created and, surprisingly, will persist for many minutes in a still pool. The dimpled depressions are typically of the order of a few millimeters in depth, but the zone of circulation around each rotation axis typically extends over a disc of some 10 to 30 centimeters radius, depending on the plate diameter. The "stationary" configuration, or coherent topological defect structure, has been defined as the Falaco Soliton. For purposes of illustration , the vertical depression has been greatly exaggerated in Figures 3 and 4.

If a thin broom handle or a rod is placed vertically in the pool, and the Falaco soliton pair is directed in its translation motion to intercept the rod symmetrically, as the soliton pair comes within range of the scattering center, or rod, (the range is approximately the separation distance of the two rotation centers) the large black spots at first shimmer and then disappear. Then a short time later, after the soliton has passed beyond the interaction range of the scattering center, the large black spots coherently reappear, mimicking the numerical simulations of soliton coherent scattering. For hydrodynamics, this observation firmly cements the idea that these objects are truly coherent "Soliton" structures. This experiment is the only (known to this author) macroscopic visual experiment that demonstrates these coherence features of soliton scattering.

If the string connecting the two endcaps is sharply "severed", the confined, two dimensional endcap singularities do not diffuse away, but instead disappear almost explosively. The process of "severing" can be accomplished by moving your hand (held under the water approximately above the circular arc or "string" connecting the two dimple vertices) in a karate chop motion. It is this observation that leads to the statement that the Falaco soliton is the macroscopic topological equivalent of the illusive hadron in elementary particle theory. The two 2-dimensional surface defects (the quarks) are bound together by a string of confinement, and cannot be isolated. The dynamics of such a coherent structure is extraordinary, for it is a system that is globally stabilized by the presence of the connecting 1-dimensional string.

For a movie of the process see [23].

2. Bifurcation Process and the Production of Topological Defects

2.1. Lessons from the bifurcation to Hopf Solitons

2.1.1. Local Stability

Consider a dynamical system that can be encoded (to within a factor, $1/\lambda$) on the variety of independent variables $\{x, y, z, t\}$ in terms of a 1-form of Action:

$$A = \{A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt\}/\lambda(x, y, z, t). \tag{2.1}$$

Then construct the Jacobian matrix of the (covariant) coefficient functions:

$$[\mathbb{J}_{jk}(A)] = \left[\partial(A_j/\lambda)/\partial x^k\right]. \tag{2.2}$$

This Jacobian matrix can be interpreted as a projective correlation mapping of "points" (contravariant vectors) into "hyperplanes" (covariant vectors). The correlation mapping is the dual of a collineation mapping, $\left[\mathbb{J}(\mathbf{V}^k)\right]$, which takes points into points. Linear (local) stability occurs at points where the (possibly complex) eigenvalues of the Jacobian matrix are such that the real parts are not positive. The eigenvalues, ξ_k, are determined by solutions to the Cayley-Hamilton characteristic polynomial of the Jacobian matrix, $[\mathbb{J}(A)]$:

$$\Theta(x,y,z,t;\ \xi) = \xi^4 - X_M\xi^3 + Y_G\xi^2 - Z_A\xi + T_K \Rightarrow 0. \tag{2.3}$$

The Cayley-Hamilton polynomial equation defines a family of implicit functions in the space of variables, $X_M(x,y,z,t)$, $Y_G(x,y,z,t)$, $Z_A(x,y,z,t)$, $T_K(x,y,z,t)$. The functions X_M, Y_G, Z_A, T_K are defined as the similarity invariants of the Jacobian matrix. If the eigenvalues, ξ_k, are distinct, then the similarity invariants are given by the expressions:

$$X_M = \xi_1 + \xi_2 + \xi_3 + \xi_4 = Trace\left[\mathbb{J}_{jk}\right], \tag{2.4}$$
$$Y_G = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + \xi_4\xi_1 + \xi_4\xi_2 + \xi_4\xi_3, \tag{2.5}$$
$$Z_A = \xi_1\xi_2\xi_3 + \xi_4\xi_1\xi_2 + \xi_4\xi_2\xi_3 + \xi_4\xi_3\xi_1, \tag{2.6}$$
$$T_K = \xi_1\xi_2\xi_3\xi_4 = \det\left[\mathbb{J}_{jk}\right]. \tag{2.7}$$

In the differential geometry of 3-dimensional space, $\{x, y, z\}$, when the scaling coefficient is chosen to be the quadratic isotropic Holder norm of index 1 (the Gauss map), then the determinant of the 3x3 Jacobian matrix vanishes, and the resulting similarity invariants become related to the mean curvature and the Gauss curvature of the Shape matrix.

Bifurcation and singularity theory involves the zero sets of the similarity invariants, and the algebraic intersections of the implicit hypersurfaces so generated by these zero sets. Recall that the theory of linear (local) stability requires that the eigenvalues of the Jacobian matrix have real parts which are not greater than zero. For a 4th order polynomial, either all 4 eigenvalues are real; or, two eigenvalues are real, and two eigenvalues are complex conjugate pairs; or there are two distinct complex conjugate pairs. Local stability therefor requires:

Local Stability

$$\text{Odd } X_M \leq 0, \qquad \text{Odd } Z_A \leq 0, \tag{2.8}$$
$$\text{Even } Y_G \geq 0, \qquad \text{Even } T_K \geq 0. \tag{2.9}$$

2.1.2. The Hopf Map

The Hopf map is a rather remarkable projective map from 4 to 3 (real or complex) dimensions that has interesting and useful topological properties related to links and braids and other forms of entanglement. As will be demonstrated, the Hopf map satisfies the criteria of Local Stability, and yet is not an integrable system, and admits irreversible dissipation. The map can be written as $\{x, y, z, s = ct\} \Rightarrow \{x1, x2, x3\}$

$$\textbf{Hopf Map} \quad |\mathbf{H}1\rangle = [x1, x2, x3]^T = [2(xz+ys), 2(xs-yz), (x^2+y^2)-(z^2+s^2)]^T. \tag{2.10}$$

A remarkable feature of this map is that

$$\langle\mathbf{H}1| \cdot |\mathbf{H}1\rangle = (x1)^2 + (x2)^2 + (x3)^2 = (x^2 + y^2 + z^2 + s^2)^2. \tag{2.11}$$

Hence a real (and imaginary) 4 dimensional sphere maps to a real 3 dimensional sphere. If the functions $[x1, x2, x3]$ are defined as $[x1/ct, x2/ct, x3/ct]$, then the 4D sphere $(x^2 + y^2 + z^2 + s^2)^2 = 1$, implies that the Hopf map formulas are equivalent to the 4D light cone. The Hopf map can also be represented in terms of complex functions by a map from C2 to R3, as given by the formulas:

$$\mathbf{H}1 = [x1, x2, x3] = [\alpha \cdot \beta^* + \beta \cdot \alpha^*, i(\alpha \cdot \beta^* - \beta \cdot \alpha^*), \alpha \cdot \alpha^* - \beta \cdot \beta^*]. \tag{2.12}$$

By permuting the formulas it is possible to construct 3 linearly independent Hopf vectors, all of which have same euclidean norm. Note that it is possible to construct complex isotropic spinors by complexifying the Hopf vectors and their permutations:

$$\text{Spinor} \quad |\sigma 1\rangle = |H2\rangle + i\,|H3\rangle\,, \qquad \langle\sigma 1| \circ |\sigma 1\rangle = 0\ . \tag{2.13}$$

It should be expected that there is a connection to surfaces of zero mean curvature and Spinors.

For $\mathbf{H}1$, the 4 independent 1 forms are given by the expressions (where $\Lambda(x, y, z, s)$ is an arbitrary scaling function):

$$d(x1) = 2zd(x) + 2sd(y) + 2xd(z) + 2yd(s) \tag{2.14}$$
$$d(x2) = 2sd(x) - 2zd(y) - 2yd(z) + 2xd(s) \tag{2.15}$$
$$d(x3) = 2xd(x) + 2yd(y) - 2zd(z) - sd(s) \tag{2.16}$$

$$A = \{-yd(x) + xd(y) - sd(z) + zd(s)\}/\Lambda. \tag{2.17}$$

The formula for the, A, 1-form can be generalized to include constant coefficients of polarization and chirality, to read

$$A_{Hopf} = \{a(-yd(x) + xd(y)) + b(-sd(z) + zd(s))\}/\Lambda. \tag{2.18}$$

It is some interest to examine the properties of the 1-form, A_{Hopf},defined hereafter as the canonical Hopf 1-form. The Jacobian matrix (for $\Lambda = 1$) becomes

$$JAC_{Hopf} := \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix} \tag{2.19}$$

with eigen vectors $(e1, e2, e3, e4)$ and eigenvalues $(+ia,\ -ia,\ +ib,\ -ib)$. The similarity invariants are:

$$\text{Odd}\quad X_M = 0, \qquad Z_A = 0. \tag{2.20}$$

$$\text{Even}\quad Y_G = a^2 + b^2 \geq 0, \qquad T_K = a^2 b^2 \geq 0. \tag{2.21}$$

Hence the canonical Hopf 1-form, A_{Hopf}, is locally stable. It the 1-form is scaled by the factor, $1/\sqrt{(x^2 + y^2 + z^2 + s^2)}$, then the similarity invariants and in all cases represents an imaginary minimal surface. The curvatures are pure imaginary, but the Gauss curvature is positive! For the simple case where $b = 0$, the Hopf map describes an minimal surface with imaginary individual curvatures. The classic real minimal surface has a Gauss curvature Y_G which is negative, and for which the individual curvatures are real.

For $\Lambda = 1$, it follows that the Hopf 1-form is of Pfaff dimension 4, and has a topological torsion 4-vector proportional to the ray vector from the origin to a point in the space,

$$\mathbf{T}_4 = -2ab[x, y, z, t]\ . \tag{2.22}$$

Any process that evolves with a component in the direction of $\mathbf{T}_4$ is thermodynamically *irreversible*, as

$$L(\mathbf{T}_4)A = -8ab\ A = Q, \tag{2.23}$$

$$\text{and } Q\hat{}dQ \neq 0. \tag{2.24}$$

Consider the Falaco Solitons to be represented by a dynamical system topologically equivalent to an exterior differential system of 1-forms,

$$\omega^k = dx^k - \mathbf{V}^k(x, y, z, t)dt \Rightarrow 0. \tag{2.25}$$

When all three 1-forms vanish, imposing the existence of a topological limit structure on the base manifold of 4 dimensions, $\{x, y, z, t\}$, the result is equivalent to a 1D solution manifold defined as a kinematic system. The solution manifold to the dynamical system is in effect a parametrization of the parameter t to the space curve $C_{parametric}$ in 4D space, where for kinematic perfection, $[\mathbf{V}^k, 1]$ is a tangent vector to the curve $C_{parametric}$. Off the kinematic solution submanifold, the non-zero values for the 1-forms, ω^k, can be interpreted as topological fluctuations from "kinematic perfection".

If "kinematic perfection" is not exact, then the three 1-forms ω^k are not precisely zero, and have a finite triple exterior product that defines a N-1=3 form in the 4 D space. From the theory of exterior differential forms it is the intersection of the zero sets of these three hypersurfaces ω^k that creates an implicit curve $C_{implicit}$ in 4D space.

$$\begin{aligned} C_{implicit} &= \omega^x\hat{}\omega^y\hat{}\omega^z & (2.26)\\ &= dx\hat{}dy\hat{}dz - \mathbf{V}^x dy\hat{}dz\hat{}dt + \mathbf{V}^y dx\hat{}dz\hat{}dt - \mathbf{V}^z dx\hat{}dy\hat{}dt & (2.27)\\ &= -i([\mathbf{V}, 1])\Omega_4. & (2.28) \end{aligned}$$

The discussion brings to mind the dualism between points (rays) and hypersurfaces (hyperplanes) in projective geometry.

If a ray (a point in a the projective 3 space of 4 dimensions) is specified by the 4 components of a the 4D vector $[\mathbf{V}, 1]$ multiplied by any non-zero factor, κ, (such that $[\mathbf{V}, 1] \approx \kappa[\mathbf{V}, 1]$), then the equation of a dual projective hyperplane is given by the expression $[\mathbf{A}, -\phi]$ such that

$$\langle\gamma[\mathbf{A}, -\phi]| \circ |\kappa[\mathbf{V}, 1]\rangle = 0. \tag{2.29}$$

The principle of projective duality [11] implies that (independent from the factors γ and κ)

$$\phi = \mathbf{A} \circ \mathbf{V}. \tag{2.30}$$

A particularly easy choice is to assume that (to within a factor)

$$\begin{aligned}\mathbf{A}_k &= \mathbf{V}^k, \text{ and } \quad \phi = \mathbf{V} \circ \mathbf{V}, &(2.31)\\ A &= V_k dx^k - V_k V^k dt. &(2.32)\\ V_k(x,y,z,t) &\equiv V^k(x,y,z,t), \text{ the 3 functions of a dynamical system.} &(2.33)\end{aligned}$$

It should be remembered that not all dynamic features are captured by the similarity invariants of a dynamic system . The antisymmetric features of the dynamics is better encoded in terms of Cartan's magic formula. Cartan's formula expresses the evolution of a 1-form of Action, A, in terms of the Lie differential with respect to a vector field, V, acting on the 1-form that encodes the properties of the physical system. For example, consider the 1-form of Action (the canonical form of a Hopf system) given by the equation

$$A_{Hopf} = \alpha(ydx - xdy) + \beta(tdz - zdt). \tag{2.34}$$

The Jacobian matrix of this Action 1-form has eigenvalues which are solutions of the characteristic equation,

$$\Theta(x,y,z,t;\ \xi)_{Hopf} = (\xi^2 + \alpha)(\xi^2 + \beta) \Rightarrow 0. \tag{2.35}$$

The eigenvalues are two conjugate pairs of pure imaginary numbers, $\{\pm i\alpha, \pm i\beta\}$ and are interpreted as "oscillation" frequencies. The similarity invariants are $X_M = 0$, $Y_G = \alpha^2 + \beta^2 > 0$, $Z_A = 0$, $T_K = \alpha^2\beta^2 > 0$. The Hopf eigenvalues have no real parts that are positive, and so the Jacobian matrix is locally stable. The criteria for a double Hopf oscillation frequency requires that the algebraically odd similarity invariants vanish and the algebraically even similarity invariants are positive definite. The stability critical point of the Hopf bifurcation occurs when all similarity invariants vanish. In such a case the oscillation frequencies are zero. This Hopf critical point is NOT necessarily the same as the thermodynamic critical point, as exhibited by a van der Waals gas. The oscillation frequencies have led the Hopf solution to be described as a "breather". The Hopf system is a locally stable system in four dimensions. Each of the pure imaginary frequencies can be associated with a "minimal" hypersurface. .

Suppose that $\beta = 0$. Then the resulting characteristic equation represents a "minimal surface" as $X_M = 0$, but with a Gauss curvature which is positive definite, $Y_G = \alpha^2 > 0$. The curvatures of the implicit surface are imaginary. In differential geometry, where the eigenfunctions can be put into correspondence with curvatures, the Hopf condition, $X_M = 0$, for a single Hopf frequency would be interpreted as "strange" minimal surface (attractor). The surface would be strange for the condition $Y_{G(hopf)} = \alpha^2 > 0$ implies that the Gauss curvature for such a minimal surface is positive. A real minimal surface has curvatures which are real and opposite in sign, such that the Gauss curvature is negative.

As a real minimal surface has eigenvalues with one positive and one negative real number, the criteria for local stability is not satisfied for real minimal surfaces. Yet experience indicates that soap films can occur as "stationary states" when stabilized by certain boundary conditions. The implication is that soap films can be globally stabilized, even though they are locally unstable.

As developed in the next section, the Falaco critical point and the Hopf critical point are the same: all similarity invariants vanish. For the autonomous examples it is possible to find an implicit surface, $Y_{G(hopf)} = Y_{G(falaco)} = 0$, in terms of the variables $\{x, y, z;\ A, B, C...\}$ where $A, B, C...$ are the parameters of the dynamical system.

Recall that the classic (real) minimal surface has real curvatures with a sum equal to zero, but with a Gauss curvature which is negative ($X_M = 0, Y_G < 0$). Such a system is not locally stable, for there exist eigenvalues of the Jacobian matrix with positive real parts. Yet persistent minimal soap films between boundaries can exist under such conditions and are apparently stable macroscopically (globally). This experimental evidence can be interpreted as an example of global stability overcoming local instability.

2.2. The bifurcation to Falaco Solitons

Similar to and guided by experience with the Hopf bifurcation, the bifurcation that leads to Falaco Solitons must agree with the experimental observation that the endcaps have negative Gauss curvature, and are in rotation. The stability of the Falaco Soliton is global, experimentally, for if the singular thread connecting the vertices is cut, the system decays non-diffusively. Hence the bifurcation to the Falaco Soliton can not imply local stability. This experimental result is related to the theoretical confinement problem in the theory of quarks. To analyze the problem consider the case where the T_K term in the Cayley-Hamilton polynomial vanishes (implying that one eigenvalue of the 4D Jacobian matrix is zero). Experience with the Hopf bifurcation suggests that Falaco Soliton may be related to another form of the characteristic polynomial, where $X_M = 0,\ Z_A = 0,\ Y_G < 0$. This bifurcation is not equivalent to the Hopf bifurcation, but has the same critical point, in the sense that all similarity invariants vanish at the critical point. Similar to the Hopf bifurcation this new bifurcation scheme can be of Pfaff topological dimension 4, which implies that the abstract thermodynamic system generated by the 1-form (which is the projective dual to the dynamical system) is an open, non-equilibrium thermodynamic system. The odd similarity invariants of the 4D Jacobian matrix must vanish. However there are substantial differences between the bifurcation that lead to Hopf solitons (breathers) and Falaco solitons. Experimentally, the Falaco soliton appears to have a projective cusp at the critical point (the vertex of the dimple) and that differs from the Hopf bifurcation which would be expected to have a projective parabola at the critical point.

When $T_K = 0$, the resulting cubic factor of the characteristic polynomial will have 1 real eigenvalue, b, one eigen value equal to zero, and possibly 1 pair of complex conjugate eigenvalues, $(\sigma + i\Omega), (\sigma - i\Omega)$. To be stable globally it is presumed that

Global Stability

$$\text{Odd } X_M = b + 2\sigma \leq 0, \quad \text{Odd } Z_A = b(\sigma^2 + \Omega^2) \leq 0, \tag{2.36}$$
$$\text{Even } Y_G = \sigma^2 + \Omega^2 + 2b\sigma \text{ undetermined}, \quad \text{Even } T_K = 0 \tag{2.37}$$

If all real coefficients are negative then $Y_G > 0$, and the system is locally stable. Such is the situation for the Hopf bifurcation. However, the Falaco Soliton experimentally requires that $Y_G < 0$.

By choosing $b \leq 0$, in order to satisfy $Z_A \leq 0$, leads to the constraint that $\sigma = -b/2 > 0$, such that the real part of the complex solution is positive, and represents an expansion, not a contraction. Substitution into the formula for Y_G leads to the condition for generation of a Falaco Soliton:

$$Y_{G(falaco)} = \Omega^2 - 3b^2/4 < 0. \tag{2.38}$$

It is apparent that local stability is lost for the complex eigenvalues of the Jacobian matrix can have positive real parts, $\sigma > 0$. Furthermore it follows that $Y_G < 0$ (leading to negative Gauss curvature) if the square of the rotation speed, Ω, is smaller than the 3/4 of the square of the real (negative) eigen value, b. This result implies that the "forces" of tension overcomes the inertial forces of rotation. In such a situation, a real minimal surface is produced (as visually required by the Falaco soliton). The result is extraordinary for it demonstrates a global stabilization is possible for a system with one contracting direction, and two expanding directions coupled with rotation. The contracting coefficient b (similar to a spring constant) is related to the surface tension in the "string" that connects the two global endcaps of negative Gaussian curvature. The critical point occurs when $\Omega^2 = 3b^2/4$.

It is conjectured that if the coefficient b is in some sense a measure of of a reciprocal length (such that $b \approx 1/R$, a curvature), then there are three interesting formulas comparing angular velocity (orbital period) and length (orbital radius).

$$\text{Falaco}: \quad \Omega^2 R^2 = \text{constant} \tag{2.39}$$

$$\text{Kepler}: \quad \Omega^2 R^3 = \text{constant} \tag{2.40}$$

$$\text{Planck}: \quad \Omega^2 R^4 = \text{constant}. \tag{2.41}$$

The bifurcations to Hopf Solitons suggest oscillations of expansions and contractions of imaginary minimal surfaces (or Soliton concentration breathers) and have been exhibited in the certain chemical reactions. On the other hand, the bifurcations to Falaco Solitons suggest the creation of spiral concentrations, or density waves, on real rotating minimal surfaces.

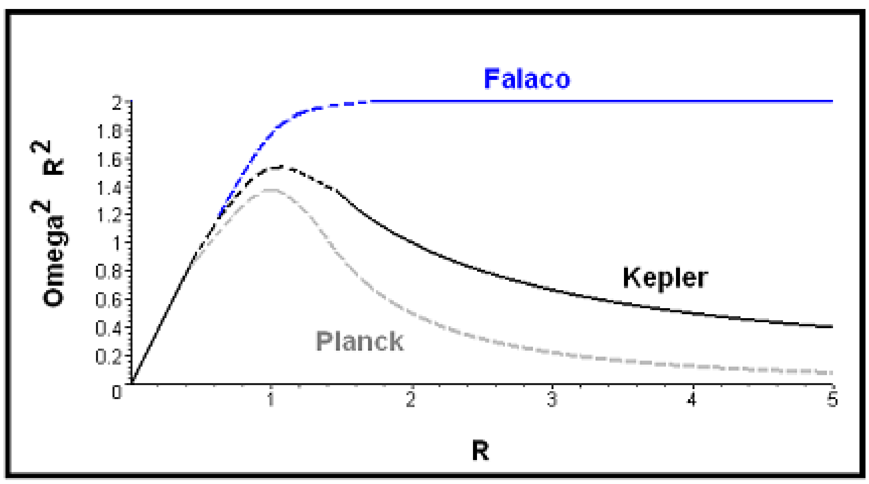


Figure 6. Spiral galaxy mass distributions

The molal density distributions (or order parameters) are complex. The visual bifurcation structures of the Falaco Solitons in the swimming pool would appear to offer an explanation as to the origin of (≈ flat) spiral arm galaxies at a cosmological level, and would suggest that the spiral arm galaxies come in pairs connected by a topological string. Moreover, the kinetic energy of the stars far from the galactic center would not vary as the radius of the "orbit" became very large. This result is counter to the Keplerian result that the kinetic energy of the stars should decrease as 1/R. If is assumed that the density distribution of star mass is more or less constant over the central region of the spiral arm flat disc-like structures, then over this region, the Newtonian gravitation force would lead to a "rigid body" result, $\Omega^2 R^2 = R$. Figure 6. demonstrates the various options:

3. Falaco Solitons in exact solutions to the Navier-Stokes equations.

The idea that multiple parameter Dynamical Systems can produce tertiary bifurcations was studied by Langford [9]. It is remarkable that these tertiary bifurcations can be demonstrated to be solutions of the Navier-Stokes equations in a rotating frame of reference [19]. Langford was interested in how these "normal" forms of dynamical systems could cause bifurcations to Hopf breather-solitons. Herein, it is also of interest to determine how and if these dynamical systems can cause bifurcations to Falaco rotational solitons.

3.1. Minimal Surface Hopf and Falaco Bifurcations

The utility of Maple becomes evident when generalizations of the Langford systems can be studied.

The generalized Langford dynamical system

$$
\begin{aligned}
f &= A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2) && (3.1)\\
g &= G + Cz && (3.2)\\
dx/dt &= \mathbf{V}^x = x(G + Cz) \mp \Omega y && (3.3)\\
dy/dt &= \mathbf{V}^y = y(G + Cz) \pm \Omega x && (3.4)\\
dz/dt &= \mathbf{V}^z = A + Bz + Fz^2 + Ez^3 + D(x^2 + y^2) && (3.5)
\end{aligned}
$$

An especially interesting case is given by the system

$$
\begin{aligned}
f &= A + P\sinh(\alpha z) + D(x^2+y^2) && (3.6)\\
g &= G + Cz && (3.7)\\
dx/dt &= \mathbf{V}^x = x(G+Cz) \mp \Omega y && (3.8)\\
dy/dt &= \mathbf{V}^y = y(G+Cz) \pm \Omega x && (3.9)\\
dz/dt &= \mathbf{V}^z = A + P\sinh(\alpha z) + D(x^2+y^2) && (3.10)
\end{aligned}
$$

Similarity Invariants for the 1-form: $A = V_k dx^k - V^k V_k dt$

$$
\begin{aligned}
X_M &= 2(G+Cz) + \alpha P\cosh(\alpha z) && (3.11)\\
Yg &= +\Omega^2 - 2CD(x^2+y^2) + (G+Cz)^2 + 2(G+Cz)P\alpha\cosh(\alpha z)\\
Z_A &= (+\Omega^2 + (G+Cz)^2)P\alpha\cosh(\alpha z) - 2CD(G+Cz)(x^2+y^2)\\
T_K &= 0 && (3.12)
\end{aligned}
$$

The similarity invariants are polarization invariants relative to the rotation parameter Ω. The criteria for Hopf oscillations requires that $X_M = 0$, and $Z_A = 0$. When these constraints are inserted into the formula for Y_G they yield $Y_{G(hopf)}$. The criteria for oscillations (and breathers) is that $Y_{G(hopf)} > 0$.

$$
\begin{aligned}
\text{Hopf Constraint :} &\quad Y_{G(hopf)} = 3\Omega^2 - 1/4\alpha^2 P^2(\cosh(\alpha z))^2 > 0 && (3.13)\\
\text{Oscillation frequencies :} &\quad \omega = \pm\sqrt{-Y_{G(hopf)}} && (3.14)
\end{aligned}
$$

Note that (again) $Y_{G(hopf)}$ is a quadratic form in terms of the rotation parameter. It is therefor easy to identify the tension parameter for the Falaco Soliton by evaluating the Falaco formula

$$
\begin{aligned}
Y_{G(falaco)} &= \Omega^2 - 3b^2/4. && (3.15)\\
\text{Falaco tension } b^2 &= (\alpha^2 P^2(\cosh(\alpha z))^2)/3). && (3.16)
\end{aligned}
$$

In this case the tension is again to be associated with a non-linear spring with extensions in the z direction.

$$\text{Helicity} = \mathbf{V} \circ curl\ \mathbf{V}$$

$$H_{bifurcation} = -\{C(x^2+y^2) + 2(A + P\sinh(\alpha z))\}\Omega.$$

If the process described by the dynamical system is to be reversible in a thermodynamic sense, then the Helicity must vanish. This constraint fixes the value of the rotation frequency Ω in the autonomous system for reversible bifurcations.

The Hopf-Falaco critical point in similarity coordinates can be mapped to an implicit surface in xyz coordinates, eliminating the rotation parameter, Ω.

$$Y_{G(hopf\_critical)} = Y_{G(falaco\_critical)} = -\{3DC(x^2+y^2) + \alpha^2 P^2(\cosh(\alpha z))^2\} \Rightarrow 0. \tag{3.17}$$

When the parameters DC have a product which is negative, then the critical surface is the catenoid – *A Minimal Surface*. That is the Hopf critical surface representation of the Gaussian curvature is an implicit surface of given by the equation,

$$(x^2+y^2) = \{(\alpha^2 P^2)/(3|DC|)\}(\cosh(\alpha z))^2 \tag{3.18}$$

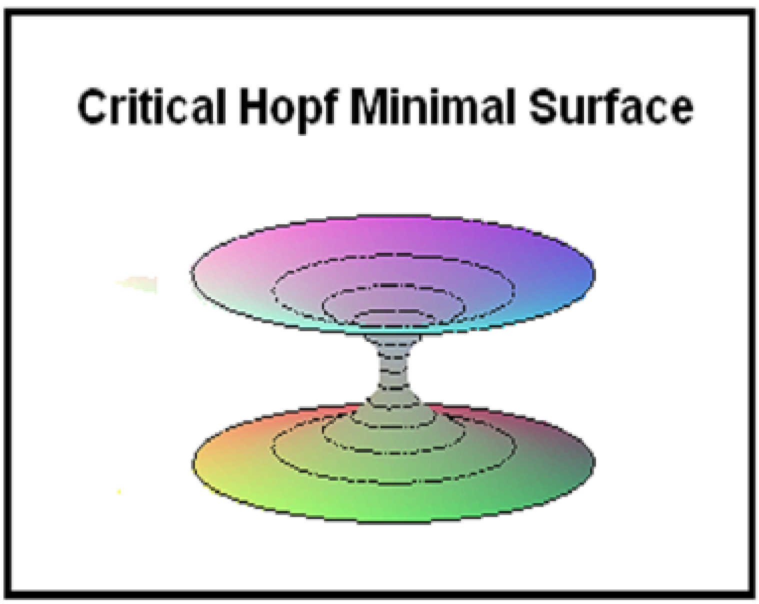


Figure 7. Surface of zero Gauss curvature at the critical point

The catenoid throat diameter is equal to the coefficient $\sqrt{(\alpha^2 P^2)/(3|DC|)}$.

3.2. Maximal Surfaces

Maximal surfaces are 2D surfaces of zero mean curvature that are generated by immersive maps from a two dimension space into a 3 dimensional space with a Lorentz metric [5]. The maximal surface is defined in terms of a space like immersion with positive Gauss curvature and with zero mean curvature. Such maximal surfaces are to be compared to minimal surfaces in a space with a Euclidean metric, but note that minimal surfaces in Euclidean space have negative Gauss curvature. Maximal surfaces can admit isolated, or "conical", singularities, where Minimal surfaces do not. Maximal surfaces can mimic catenoidal and helical surfaces of Euclidean theory, but may exhibit singular subsets of points. It is remarkable (and discussed in the next section) that such maximal surfaces can appear in fluids as propagating long lived topological defects which have been described above as Falaco Solitons.

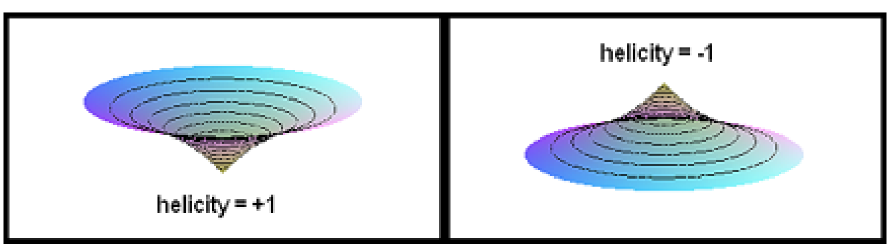
Consider a 3D space with a Minkowski - Lorentz metric of the form

$$(ds)^2 = (dx)^2 + (dy)^2 - (dz)^2. \tag{3.19}$$

The immersion

$$R(u, v) = [(\sinh(v)\cos(u), (p\sinh(a)\sin(u), h\ v] \tag{3.20}$$

generates a surface of zero mean curvature in a space with a Minkowski metric. The coefficient p is related to the handedness of the rotation about the z axis, and h is related to the helicity along the z axis. The surface is of zero mean curvature, but the metric vanishes at the conical singular point: the Gauss curvature becomes infinite. The immersion does not generate a minimal surface in euclidean space. For Other examples of zero mean curvature surfaces in both Euclidean and Minkowski spaces see [10]



Figures 8a. and 8b. Minkowski surfaces of zero mean curvature

The surface is similar to the hyperbolic minimal surface (Catenoid) in Euclidean geometry, but here, unlike the Euclidean catenoid, the Minkowski catenoid has a singular point. The surface is sensitive to the sign of the directional chirality ($h = \pm 1$), but is not sensitive to the handedness of polarization, p.

3.2.1. Immersions that do not depend upon the 3D signature

The hyperbolic rotational immersion,

$$R(u, v) = [\cosh(v)\cos(u), p\cosh(v)\sin(u), hv], \tag{3.21}$$

generates a minimal surface of zero mean curvature both in a space with an Euclidean metric, or in a space with a Minkowski metric. The surface "mimics" a Wheeler wormhole, and the soap film between two rings separated by a diameter. The zero mean curvature surface is also sensitive to the sign of the directional chirality ($h = \pm 1$), but is not sensitive to the handedness of rotational polarization, p.

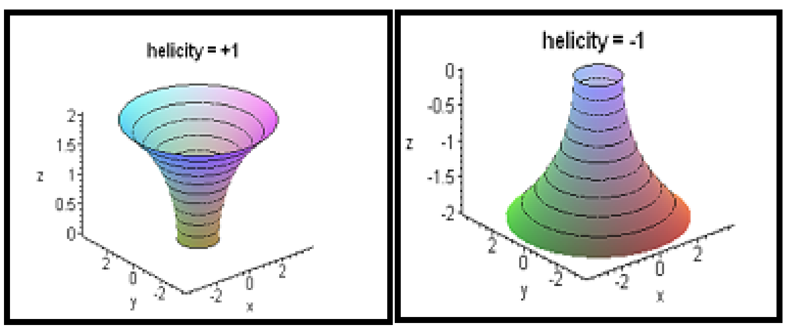


Fig. 9a and 9b. Rotational Surfaces zero mean curvature independent of 3D signature

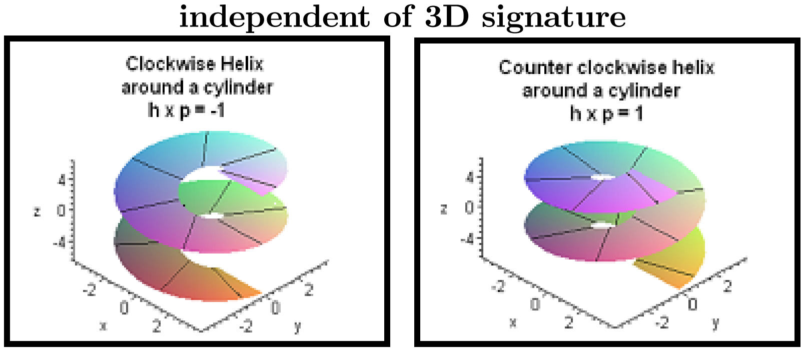


Figure 10a. and 10b. Helical Surfaces of zero mean curvature independent from 3D signature.

On the other hand, the surface generated by the hyperbolic helical immersion

$$R(u,v) = [(\cosh(v)\cos(u), (p\cosh(v)\sin(u), h\ u] \tag{3.22}$$

also is a surface of zero mean curvature in *both* a Euclidean space or in Minkowski space.

The surfaces are ruled helices rapped around a "hole" of radius unity. The Helical surface is sensitive to the sign of the product of the rotational polarization, p, and the directional chirality, h. The Gauss curvature of the immersion is negative and bounded in Euclidean Space. The Gauss curvature of the immersion is positive and singular for $v = 0$ in the Minkowski space. Both surfaces have zero mean curvature. The principal surface curvatures are real and of opposite in sign for the Euclidean 3 space, and are pure imaginary and of opposite sign in Minkowski 3 space. In both cases the Gauss curvature is real but of different signs.

The zero mean curvature surfaces, with a singular point (as in Figures 8a. and 8b.), can be formed experimentally in a fluid. The experimental evidence is presented below. The idea that 3-dimensional space may or may not be visually Euclidean challenges a dogmatic precept of modern physics, where it is rarely perceived that physical 3D space can be anything but Euclidean. However, as discussed in the following section, the occurrence of long lived rotational structures in the free surface of a water, which have been described as Falaco Solitons, exhibit the features of maximal surfaces in a Lorentz - Minkowski space. The Falaco Solitons are topological defect structures easily replicated in an experimental sense. Optical measurements indicate that the surface defect structures have a zero mean curvature. In addition, the surface defect structures have an apparent conical singularity which is an artifact of the signature of a maximal space-like surface in Minkowski space. Maximal surfaces are generated by immersive maps from a two dimension space into a 3 dimensional space with a Lorentz metric [5]. The maximal surface is defined in terms of a space like immersion with positive Gauss curvature and with zero mean curvature. Such surfaces are related to minimal surfaces in a space with a Euclidean metric, but minimal surfaces in Euclidean space have negative Gauss curvature. Maximal Surfaces can admit isolated, or "conical", singularities, where Minimal surfaces can not. Maximal surfaces can mimic catenoidal and helical surfaces of Euclidean theory, but may exhibit singular subsets of points.

The zero mean curvature surfaces, with a singular point, can be formed experimentally in a fluid. The experimental evidence is given by the existence of the Falaco Solitons. The idea that 3-dimensional space may or may not be Euclidean challenges a dogmatic precept of modern physics, where it is rarely perceived that physical 3D space can be anything but Euclidean. However, as discussed above, the occurrence of long lived rotational structures in the free surface of a water, which have been described as Falaco Solitons, exhibit the features of maximal surfaces in a Lorentz - Minkowski space. The Falaco Solitons are topological defect structures easily replicated in an experimental sense. Optical measurements indicate that the surface defect structures have a zero mean curvature. In addition, the surface defect structures have an apparent conical singularity which is an artifact of the signature of a maximal space-like surface in Minkowski space.

The conjecture is that the Falaco Solitons are topological defects caused by the decay of a dissipative Pfaff dimension 4 domain, with a spacelike Euclidean structure, followed by a topological bifurcation process that changes the space-like Sylvester signature from a 3D Euclidean structure to a space like 3D Minkowski structure.

4. Landau Ginsburg structures, Falaco Solitons and Spiral Arm structures

The Falaco experiments lead to the idea that such topological defects are available at all scales, and are related to fourth order Landau - Ginsburg structures. Figure 11 is adapted from Tornkvist [26].

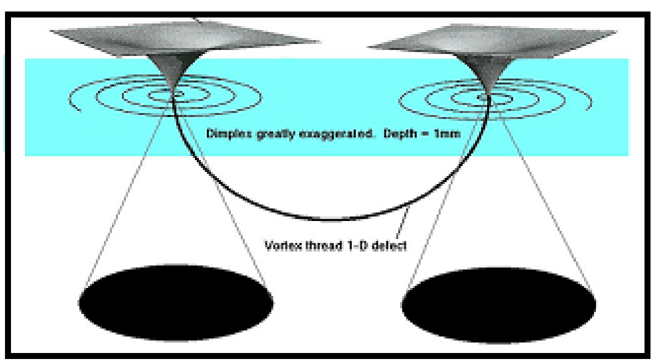


Figure 11. Falaco Solitons related to Landau Ginsburg theory

The Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, if the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces. For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, A, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [26].

Figure 12. Hubble photo: Cosmic strings linking spiral arm galaxies?

In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. Based on the experimental creation of Falaco Solitons in a swimming pool, it has been conjectured that M31 and the Milky Way galaxies could be connected by a topological defect thread [29]. Only recently has photographic evidence appeared suggesting that Galaxies may be connect by strings.

4.1. Wheeler Wormholes and Falaco Strings between Branes

It is extraordinary, but the Falaco Solitons appear to be another form of a zero mean curvature surface structure, either related to macroscopic realizations of the Wheeler wormhole (with a very narrow throat), or to Spinor surfaces generated by complex eigen direction fields of infinitesimal rotations. The Wheeler wormhole structure was presented early on by Wheeler (1955), but was considered to be unattainable in a practical sense. To quote Zeldovich p. 126 [34]

> "The throat or "wormhole" (in a Kruskal metric) as Wheeler calls it, connects regions of the same physical space which are extremely remote from each other. (Zeldovich then gives a sketch that topologically is similar to the Falaco Soliton). Such a topology implies the existence of 'truly geometrodynamic objects' which are unknown to physics. Wheeler suggests that such objects have a bearing on the nature of elementary particles and anti particles and the relationships between them. However, this idea has not yet borne fruit; and there are no macroscopic"geometrodynamic objects" in nature that we know of. Thus we shall not consider such a possibility."

This quotation dates back to the period 1967-1971.

Now the experimental evidence justifies (again) Wheeler's intuition. Both the Wheeler wormhole and the Falaco Soliton are related to surface structures of zero mean curvature. The catenoidal surface of zero mean curvature, and negative Gauss curvature, in a 3D Euclidean space is a Wheeler Wormhole (with an open throat), while the conical surface of zero mean curvature, and positive Gauss curvature, and its conical singular point in a 3D Minkowski space is a part of the rotationally induced Falaco Soliton.

Remark 5. *The bottom line is that the remarkable features of creating a stable surface of zero mean curvature and positive Gauss curvature (the Falaco Soliton) is explained either by assuming that the usual 3D Euclidean Signature is rotationally dependent and can topologically evolve into a 3D Minkowski Signature; or, the Euclidean Signature is preserved, and a macroscopic evolutionary process described by complex Spinor direction fields (which are not the same as diffeomorphic vector fields) must be admitted on thermodynamic grounds.*

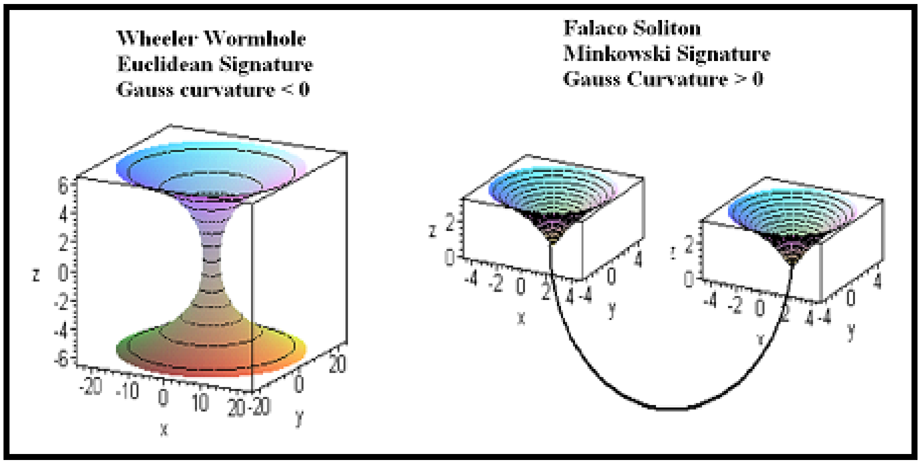


Figure 13. Rotational Surfaces of Zero mean curvature in Euclidean and Minkowski 3 space

The Falaco Soliton endcap dimples (which are presumed to be surfaces of zero mean curvature and positive Gauss curvature) are related to Spinor eigen direction fields associated with antisymmetric matrices representing Symplectic spaces. If the Maximal surfaces appear as deformations in disconnected hypersurfaces of discontinuity, the topological structure has the appearance of "strings connecting branes", a concept touted by the string theorists .

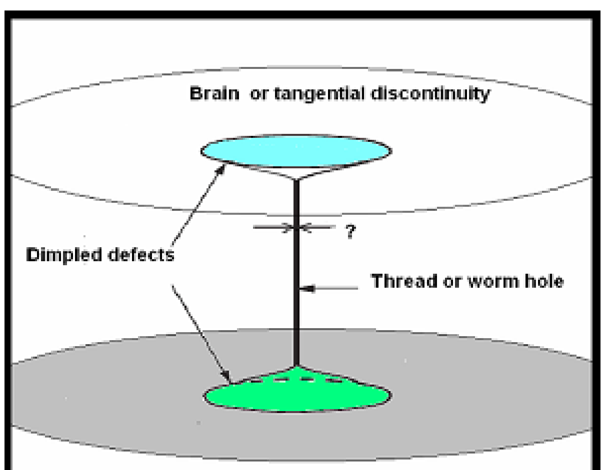


Figure 14. Falaco Solitons as connected dimples between Branes.

The new feature is that the "brane" surface of discontinuity is deformed by the Maximal surface dimple (which has been alluded to as a space-time foam [4]) This structure motivates the next section in which the idea is used to model the photon. The idea is also related to the rotational structures of rotating Bose-Einstein condensates [27].

In the next section, it is assumed that a thermodynamic (electromagnetic) system can be encoded by a 1-form of Action potentials, A, which leads by exterior differentiation to a 2-form of field intensities, $F = dA$. The null eigenvectors of the antisymmetric matrix representation of F will form 3D expanding spherical surfaces of propagating field discontinuities (related to the spatial portions of the Minkowski lightcone where $F\hat{}F = 0$). In addition, the isotropic Spinor eigenvectors of F will form surfaces of zero mean curvature as defect structures on the spherical spatial portions of the lightcone. The result is a Falaco Soliton pair (with $A\hat{}F \neq 0$) between the two bounding cycles of a spherical shell. The claim is that this concept serves as a model for the Photon.

5. Falaco Solitons as a topological model for a photon.

The idea is to combine the topological features of the Minkowski signature, the possibilities of coherent states of "stationary" topology (solitons) for non-equilibrium, but thermodynamically closed, systems of Pfaff topological dimension 3 (with $A\hat{}F \neq 0$), and the fact that for such systems the electromagnetic 2-form, $F = dA$, has one pair of eigenvectors of eigenvalue zero, and one pair of complex conjugate isotropic null eigenvector arrays with imaginary eigenvalues. The eigenvectors with zero eigenvalues form Minkowski lightcones with $F\hat{}F = 0$. Consider two causal expanding spheres (two light cones) representing the "on" and "off" propagating discontinuity defects (as expanding concentric spheres in 3D). The concentric spherical surfaces of field discontinuity bound an interior region of finite electromagnetic field intensities, $\mathbf{E}$ and $\mathbf{B}$. The conjugate pair of Spinor eigenvectors define 2D surfaces of zero mean curvature as conical topological deformation defects on the light cones.

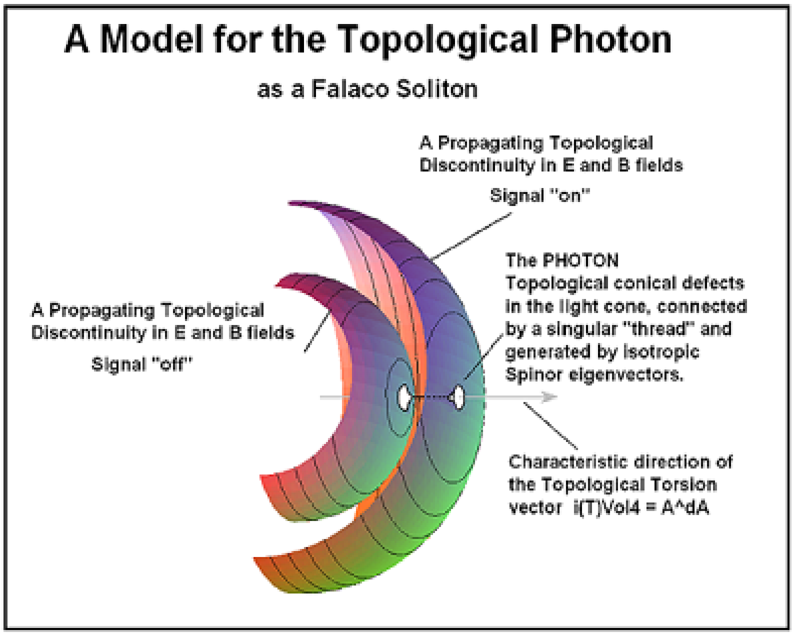


Fig. 15. The Photon as a Falaco Soliton between lightcone shells

The region interior to the two light cone "shells" is a region of non equilibrium thermodynamics, where the Topological Torsion, $A\hat{}F$, is not zero, but has zero divergence. Hence the closed integrals of the three form, $A\hat{}F$, have rational ratios. That is, they are topologically quantized. It is suggested that these "quantized" torsion concepts are related to the concept of quantized orbital angular momentum introduced in recent investigations of photon systems. (The intrinsic spin is related to the 3-form $A\hat{}G$. [31]) The conical defects on each light cone are connected by a 1D "string", or "vortex tube", of zero radius, determined by the condition that evolution, V, in the direction of the components of the 3-form, $A\hat{}F$, of topological torsion, are extremal. That is, the thermodynamic work vanishes: $W = i(V)dA = i(V)F \Rightarrow 0$.

As an example, consider the 1-form of Action given by,

$$A = (m_e/e)\{\omega(xdy - ydx) - c^2dt\}, \tag{5.1}$$

where the constants $(m_{electron}/e = h/(ec\lambda_{Compton})), \omega$ and c have been chosen on grounds of dimensional analysis. The Pfaff sequence demonstrates the the Pfaff topological dimension relative to the 1-form A is 3:

$$\begin{aligned} F &= dA = (m_e/e)2\omega\ dx\hat{}dy = B_z dx\hat{}dy, && (5.2)\\ A\hat{}F &= (m_e c^2/e)B_z dx\hat{}dy\hat{}dt, && (5.3)\\ F\hat{}F &= 0. && (5.4) \end{aligned}$$

There is no **E** field, but there is a **B** field component along the z axis. Hence the example has the properties

$$\mathbf{E} = 0,\ \mathbf{B} \neq 0,\ \mathbf{A} \neq 0,\ \ \mathbf{E} \circ \mathbf{B} = 0,\ \ \mathbf{A} \circ \mathbf{B} = 0 \qquad (5.5)$$

Note that the scalar and vector potentials are given by the expressions

$$\begin{aligned} \phi &= (m_e c^2/e), && (5.6)\\ \mathbf{A} &= (m_e/e)\omega[-y, x, 0]. && (5.7) \end{aligned}$$

The vector potential is tangent to a circle about the origin in the z = 0 plane. The direction field generated by the topological torsion vector is

$$\begin{aligned} \mathbf{T}_4 &= [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi,\ \mathbf{A} \circ \mathbf{B}] && (5.8)\\ &= [0, 0, B_z(m_e c^2/e), 0]. && (5.9) \end{aligned}$$

For evolutionary processes $\mathbf{V}_4$ in the direction of $\mathbf{T}_4$, it follows that the Work 1-form vanishes.

$$\begin{aligned} W &= i(\mathbf{V}_4)F = -\{\mathbf{E} + \mathbf{V} \times \mathbf{B}\} \circ d\mathbf{r} + (\mathbf{E} \circ \mathbf{V})dt, && (5.10)\\ W &= i(\mathbf{T}_4)F = -\{\mathbf{0} + \mathbf{B}\phi \times \mathbf{B}\} \circ d\mathbf{r} + (0)dt \Rightarrow 0. && (5.11) \end{aligned}$$

The evolutionary velocity field **V** in the direction of $\mathbf{T}_4$ is proportional to the **B** field.

This result gains credence from the observations of similar topological defects in fluid systems, the Falaco Solitons. Thermodynamic systems of Pfaff topological dimension 3 (based on the 1-form, A) are non equilibrium, thermodynamically closed systems that can exchange energy (radiation) but not mass with their environments . When the Photon is "created" the Pfaff topological dimension is presumed to be 4, with evolution along a space time direction field given by the

Topological Torsion vector, $\mathbf{T}_4$. The processes is thermodynamically irreversible, and $(\mathbf{E} \circ \mathbf{B}) \neq 0$. The process evolves continuously to domains of Pfaff topological dimension 3, forming the "condensations" - or Photons - of topological coherence as stationary, but excited, states of a Hamiltonian process. It is conjectured that the conical topological defects are not constrained by a limiting speed C, but can move (transversely on the light cone) with speeds given by the projective Moebius transformations, which can vary from zero to infinity.

6. Continuous Topological Evolution

The objective of this section is to examine topological aspects and defects of thermodynamic physical systems and their possible continuous topological evolution, creation, and destruction on a cosmological scale. The creation and evolution of stars and galaxies will be interpreted herein in terms of the creation of topological defects of Pfaff topological dimension 3 and evolutionary phase changes in a very dilute turbulent, non-equilibrium, thermodynamic system of maximal Pfaff topological dimension 4. The cosmology so constructed is opposite in viewpoint to those efforts which hope to describe the universe in terms of properties inherent in the quantum world of Bose-Einstein condensates, super conductors, and superfluids [27]. Both approaches utilize the ideas of topological defects, but thermodynamically the approaches are opposite in the sense that the quantum method involves, essentially, equilibrium systems, while the approach presented herein is based upon non-equilibrium systems. Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent - but not equilibrium - structures of Pfaff topological dimension 3 in this non-equilibrium system of Pfaff topological dimension 4. The topological theory of the ubiquitous van der Waals gas leads to the concepts of negative pressure, string tension, and a Higgs potential as natural consequences of a topological point of view applied to thermodynamics. Perhaps of more importance is the fact that these concepts do not depend explicitly upon the geometric constraints of metric or connection, and yield a different perspective on the concept of gravity.

The original motivation for this conjecture is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [7]. However, the methods used herein are not statistical, not quantum mechanical, and instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution [17]. Landau and Lifshitz emphasized that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a 1/r potential giving a $1/r^2$ force law of attraction. Hence, as a cosmological model, the almost perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the granularity of the night sky and for the $1/r^2$ force law ascribed to gravitational forces between for massive aggregates.

A topological (and non statistical) thermodynamic approach can be used to demonstrate how a four dimensional variety can support a turbulent, non-equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas [28]. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non-equilibrium system. For those physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton 4 dimensional characteristic polynomial of the Jacobian matrix generates a universal phase equation. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton 4 dimensional characteristic polynomial. These constraints define equivalence classes of topological properties.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which, when constrained to a minimal hypersurface, is related to a swallowtail bifurcation set. The swallowtail defect structure is homeomorphic to the Gibbs surface of a van der Waals gas. Another possible defect structure corresponds to the implicit hypersurface surface defined by a zero determinant condition imposed upon the Jacobian matrix. On 4 dimensional variety (space-time) , this non-degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non-equilibrium thermodynamic states whose kernel is a van der Waals gas. Hence this universal topological method for creating a low density turbulent non-equilibrium media leads to the setting examined statistically by Landau and Lifshitz in terms of classical fluctuations about the critical point.

The conjecture presented herein is that non-equilibrium topological defects in a non-equilibrium 4 dimensional medium represent the stars and galaxies, which are gravitationally attracted singularities (correlations of fluctuations of density fluctuations) of a real gas near its critical point. Note that the Cartan methods do not impose (*a priori.)* a constraint of a metric, connection, or gauge, but do utilize the topological properties associated with constraints placed on the similarity invariants of the universal phase function.

Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent (but not equilibrium) self-organizing structures of Pfaff topological dimension 3 formed by irreversible topological evolution in this non-equilibrium system of Pfaff topological dimension 4.

The turbulent non-equilibrium thermodynamic cosmology of a real gas near its critical point yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies.
2. The Newtonian law of gravitational attraction proportional to $1/\mathrm{r}^2$.
3. The expansion of the universe (4th order curvature effects).
4. The possibility of domains of negative pressure (explaining what has recently been called dark energy) due to a classical Higgs mechanism for aggregates below the critical temperature (3rd order curvature effects)
5. The possibility of domains where gravitational effects (2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system.
6. The possibility of cohesion properties (explaining what has recently been called dark matter) due to string or surface tension (1st order Mean curvature effects)
7. Black Holes (generated by Petrov Type D solutions in gravitational theory [3]) are to be related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.

7. Summary

As the Falaco phenomenon appears to be the result of a topological defect, it follows that as a topological property of hydrodynamic evolution, it could appear in any density discontinuity, at any scale. This rotational pairing mechanism, as a topological phenomenon, is independent from size and shape, and could occur at both the microscopic and the cosmic scales. In fact, as mentioned above, during the formative stages of the Falaco Soliton pair, the decaying Rankine vortices exhibit spiral arms easily visible as caustics emanating from the boundary of each vortex core. The observation is so striking that it leads to the conjecture: Can the nucleus of M31 be connected to the nucleus of our Milky way galaxy by a tubular cosmic thread? Can material be ejected from one galaxy to another along this comic thread? Can barred spirals be Spiral Arm galaxies at an early stage of formation - the bar being and exhibition of material circulating about the stabilizing thread? At smaller scales, the concept also permits the development of another mechanism for producing spin-pairing of electrons in the discontinuity of the Fermi surface, or in two dimensional charge distributions. Could this spin pairing mechanism, depending on transverse wave, not longitudinal wave, coupling be another mechanism for explaining superconductivity? As the defect is inherently 3-dimensional, it must be associated with a 3-form of Topological Torsion, A^dA, introduced by the author in 1976 [13] [15] [16] [20], but now more commonly called a Chern Simons term. These ideas were exploited in an attempt to explain high TC superconductivity [18]. To this author the importance of the Falaco Solitons is that they offer the first clean experimental evidence of topological defects taking place in a dynamical system. Moreover, the experiments are fascinating, easily replicated by anyone with access to a swimming pool, and stimulate thinking in almost everyone that observes them, no matter what his field of expertise. They certainly are among the most easily produced solitons.

More detail (with downloadable pdf files of almost all publications) may be found on the web site:

http://www.cartan.pair.com

The original observation was first described at a Dynamics Days conference in Austin, TX, [14] and has been reported, as parts of other research, in various hydrodynamic publications, but it is apparent that these concepts have not penetrated into other areas of research. As the phenomenon is a topological issue, and can happen at all scales, the Falaco Soliton should be a natural artifact of both the sub-atomic and the cosmological worlds. The reason d'etre for this short article is to bring this idea to the attention of other researchers who might find the concept interesting and stimulating to their own research

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